On the topological decomposition of the hypersurfaces in projective toric manifolds

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Abstract

In this paper, we want to discuss the topology of the non-singular hypersurface Y^n with complex dimension n in a projective toric manifold X^{n+1}. When n is odd, our main results are a decomposition of $Y^n \cong Y' \sharp\, s(S^n \times S^n)$ as a connected sum of s copies of $S^n \times S^n$ with a differential manifold Y' such that $b_n(Y') = 0$ or 2. When n is even and the degree of Y in X is big enough, we find that Y also admits such a decomposition $Y' \sharp\, s(S^n \times S^n)$, where Y' satisfy $b_n(Y') - |sign(Y')| = b_n(X) \pm sign(H^n(X))$, where $sign(H^n(X))$ is the signature of a certain bilinear form defined on $H^n(X, \mathbb{Z})$.

1 Introduction

1.1 Projective toric manifold and its hypersurfaces

Definition 1.1. *A toric variety is a normal algebraic variety X containing the algebraic torus $(\mathbb{C}^*)^n$ as a Zariski open subset in such a way that the normal action $(\mathbb{C}^*)^n$ on itself extends to an action on X.*

In this paper, we call X a ***projective toric manifold*** *if X is a compact, smooth toric variety that admits a holomorphic embedding into a certain $\mathbb{C}P^N$.*

The algebraic topology of projective toric manifold has been fully studied by many people and many results can be found in these two classical books [4],[5]. In this paper, what we need are the following two propositions ([5], page 56,101,102).

Proposition 1.2. *Let X be a projective toric manifold, then X is simply connected and the odd dimension homology groups of X vanish, i.e. $H_{odd}(X, \mathbb{Z}) = 0$.*

Proposition 1.3. *$H_*(X, \mathbb{Z})$ can be generated by the projective toric submanifolds of X, i.e. there exist smooth toric submanifolds $\{X_i\}$ with $x_i = [X_i] \in H_*(X, \mathbb{Z})$ such that the homomorphism*

$$\sum \mathbb{Z}x_i \longrightarrow H_*(X, \mathbb{Z})$$

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is surjective.

Then we introduce the hypersurface of a projective toric manifold. Let X be a projective toric manifold. For any holomorphic embedding $X \hookrightarrow \mathbb{C}P^N$, let F_1 be a hyperplane of $\mathbb{C}P^N$, we get a subvariety $i: Y = F_1 \cap X \hookrightarrow X$ of X and Y is called a **hypersurface** of X. By Bertini's theorem, for a generic hyperplane F_1 in $\mathbb{C}P^N$, Y is smooth.

Given such a hypersurface Y of $X \hookrightarrow \mathbb{C}P^N$, we can also construct the smooth hypersurface $i_d: Y_d \hookrightarrow X$ of X with $(i_d)_*[Y_d] = d(i_*[Y]),\ 0 < d \in \mathbb{Z}$. Indeed, we can take $Y_d := F_d \cap X$, where F_d is a generic hypersurface of $\mathbb{C}P^N$ with degree d and it is well-known that Y_d is also a smooth hypersurface of X.

In this paper, all the hypersurfaces we consider are smooth and when we say a hypersurface Y_d, it always means Y_d is a smooth hypersurface.

Similar to the degree of a hypersurface in $\mathbb{C}P^n$, we can define the degree of a (smooth) hypersurface in X. Let α_Y be the element of $H^2(X, \mathbb{Z})$ such that $\alpha_Y \cap [X] = i_*[Y]$. We define the **degree** of a hypersurface Y in X by

$$degY :=< \alpha_Y^{n+1}, [X] >$$

For the hypersurface Y_d, we have relation $d\alpha_Y = \alpha_{Y_d}$ and we have $degY_d = d^{n+1}degY$.

1.2 Main results

Let X^{n+1} be a projective toric manifold with complex dimension $n+1$, $n > 2$. Let $i: Y \hookrightarrow X$ be the hypersurface of X with complex dimension n and $i_d: Y_d \hookrightarrow X$ be the hypersurface with $(i_d)_*[Y_d] = d(i_*[Y]),\ 0 < d \in \mathbb{Z}$. In this paper, we want to discuss the topological decomposition of the hypersurface Y_d. Our main results are:

Theorem 1.4. *When n is odd, for any integer $d > 0$, we have decomposition:*

$$Y_d \cong Y_d^{'} \sharp\ s_d(S^n \times S^n)$$

where the n-th Betti number $b_n(Y_d^{'}) = 0$ or 2.

Theorem 1.5. *When n is even, for sufficiently big $d >> 0$, we have decomposition:*

$$Y_d \cong Y_d^{'} \sharp\ s_d(S^n \times S^n)$$

with $s_d = \frac{b_n(Y_d) - b_n(X) - |sign(Y_d) - sign(H^n(X))|}{2}$, here $sign(Y_d)$ is the classical signature of Y_d and $sign(H^n(X))$ is the signature of the bilinear form defined by:

$$H^n(X) \otimes H^n(X) \longrightarrow \mathbb{Z}$$

$$(x, y) \mapsto< x \cup y \cup \alpha_{Y_d}, [X] >$$

Furthermore, we have limit estimate:

$$0 < \lim_{d \to +\infty} \frac{2s_d}{degY_d} = \lim_{d \to +\infty} \frac{2s_d}{d^{n+1}degY} = 1 - 2^{n+1}(2^{n+1} - 1)\frac{B_{\frac{n+2}{2}}}{(n+1)!} < 1$$

here $B_{\frac{n+2}{2}}$ is the $\frac{n+2}{2}$-th Bernouli number.

For Y'_d, we have relations $b_n(Y'_d) = b_n(Y_d) - 2s_d$ and $sign(Y_d) = sign(Y'_d)$. By the limit estimates in proposition 4.4, we know $|sign(Y_d)| = |sign(Y'_d)|$ tends to $+\infty$ as $d \to +\infty$. From theorem 1.5, we can deduce that

Corollary 1.6. *When n is even and d is big enough:*

$$b_n(Y'_d) - |sign(Y'_d)| = b_n(X) \pm |sign(H^n(X))|$$

Remark 1.7. Let F be a nonsingular algebraic hypersurface in complex projective space, in [8], Kulkarni and Wood proved that there is a differentiable connected sum decomposition

$$F = M \sharp k(S^n \times S^n)$$

where $b_n(M) = 0$ or 2 for n odd, and $b_n(M) - |sign(M)| = b_n(\mathbb{C}P^n) \pm sign(H^n(\mathbb{C}P^{n+1})) = 0$ or 2 for n even.

Our theorem is a generalization of their theorem to the case of hypersurfaces in projective toric manifolds.

2 Basic idea of removing handles

2.1 Geometric point of view

Choose a point $(x, y) \in S^n \times S^n$ and there are two embedded spheres: $S_1 := S^n \times \{y\}$, $S_2 := \{x\} \times S^n \hookrightarrow S^n \times S^n$ with properties:
(1). S_1 intersects S_2 transversally at one point (x, y).
(2). The normal bundles of S_1, S_2 in $S^n \times S^n$ are trivial.
(3). Denote $\eta_1 := S_1 \times D^n \subset S^n \times S^n$ and $\eta_2 := S_2 \times D^n \subset S^n \times S^n$ by the closure of their normal bundles, we see $\eta_1 \cup \eta_2$ is a manifold with boundary S^{2n-1} and

$$S^n \times S^n = (\eta_1 \cup \eta_2) \cup_{S^{2n-1}} D^{2n}$$

Conversely, let M^{2n} be a smooth manifold and S_1, S_2 be two embedded n-spheres of M^{2n} with:
(1). S_1 intersects S_2 transversally at one point.
(2). The normal bundles of S_1 and S_2 are trivial.
We denote $\xi_1 := S_1 \times D^n$, $\xi_2 := S_2 \times D^n$ by the closure of their normal bundles. Observe that $\eta_1 \cup \eta_2 \cong \xi_1 \cup \xi_2$ and we get:

$$M \cong (M - \xi_1^\circ \cup \xi_2^\circ) \cup_{S^{2n-1}} (\eta_1 \cup \eta_2) \cong M' \sharp\, S^n \times S^n$$

where $M' = (M - \xi_1^\circ \cup \xi_2^\circ) \cup_{S^{2n-1}} D^{2n}$. This is the basic idea of removing handles from a $2n$-manifold ([12]). Next we want to realize this idea by algebraic topology.

2.2 Homological point of view

From the point of view of homology, let M^{2n} be a simply connected smooth closed manifold of dimension $2n$, $n > 2$ and $h : \pi_n(M) \longrightarrow H_n(M, \mathbb{Z})$ be the Hurewicz map. For every

$\alpha,\ \beta \in h(\pi_n(M)) \subset H_n(M,\mathbb{Z})$ with intersection number $\alpha\cdot\beta = 1$, by Whitney's embedding theory and Whitney's trick ([10], p142), there are two embedding n-spheres $f_\alpha,\ f_\beta : S^n \hookrightarrow M^{2n}$ with:

(1). The homology elements α and β are represented by $f_\alpha,\ f_\beta$, i.e. $(f_\alpha)_*[S^n] = \alpha,\ (f_\beta)_*[S^n] = \beta$

(2). The spheres $f_\alpha(S^n)$ and $f_\beta(S^n)$ intersect transversally at only one point.

Following the geometric idea of removing handles, the next question is how to determine the normal bundles. In general, the normal bundles of $f_\alpha(S^n),\ f_\beta(S^n)$ are not easy to determine.

In this paper, the situation seems relatively simpler: let $K \subset h(\pi_n(M))$ be a free Abel group such that each element $\alpha \in K$ can be represented by an embedded n-sphere $f_\alpha : S^n \hookrightarrow M$ with **stable trivial** normal bundle.

When n is even, for the embedding f_α representing $\alpha \in K$, the normal bundle of f_α is just determined by the self-intersection number $\alpha\cdot\alpha$ of α. Indeed, $\alpha\cdot\alpha = 0$ if and only if the normal bundle of f_α is trivial. So, if we could find a free subgroup $\oplus_{i=1}^s(\mathbb{Z}\alpha_i \oplus \beta_i)$ of K with intersection matrix $\oplus_{i=1}^s \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, topologically, M admits a decomposition:

$$M \cong M' \sharp\ s(S^n \times S^n)$$

When n is odd, the intersection number $\alpha\cdot\alpha$ is always zero and can not determine the normal bundle of f_α, we need two techniques.

Technique 1: find a quadratic function $\psi : K \longrightarrow \mathbb{Z}_2$ with:

(1). $\psi(\alpha+\beta) = \psi(\alpha) + \psi(\beta) + (\alpha\cdot\beta)_2$, where $(\alpha\cdot\beta)_2 \in \mathbb{Z}_2$ is the mod 2 class of the intersection number $\alpha\cdot\beta \in \mathbb{Z}$, which is also the definition of the quadratic function over $\mathbb{Z}$.

(2). $\psi(\alpha) = 0$ if and only if α can be represented by an embedded n-sphere f_α with trivial normal bundle.

For any free subgroup $\oplus_{i=0}^s(\mathbb{Z}\alpha_i' \oplus \beta_i')$ of K with intersection matrix $\oplus_{i=0}^s \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$, by the standard results of the quadratic function ([15], p172), we can find a new basis $\{\alpha_i, \beta_i\}$ of this subgroup such that the intersection matrix of $\mathbb{Z}\alpha_0 \oplus \mathbb{Z}\beta_0 \oplus \{\oplus_{i=1}^s(\mathbb{Z}\alpha_i \oplus \beta_i)\}$ is still $\oplus_{i=0}^s \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ and $\psi(\alpha_i) = \psi(\beta_i) = 0, i \neq 0,\ \psi(\alpha_0) = \psi(\beta_0) = 0$ or 1. In this case, although we can not determine the value of $\psi(\alpha_0)$, at least, we have decomposition:

$$M \cong M' \sharp\ s(S^n \times S^n)$$

In general, the quadratic function ψ is not always exist on K and we need the second technique.

Technique 2: find an embedded n-sphere $g : S^n \hookrightarrow M$ with:

(1). $g_*[S^n] = 0 \in H_n(M,\mathbb{Z})$.

(2). The normal bundle η_g of $g : S^n \hookrightarrow M$ is isomorphic to the tangent bundle TS^n of S^n.

If we could find such an embedding g, for every element $\alpha \in K$ which is represented by an embedding f_α, if the stable trivial normal bundle η_{f_α} is not trivial, by Wall's technique ([15], p167), there exists a new embedding f'_α with normal bundle $\eta_{f'_\alpha}$ such that:

(1). $f'_\alpha = f_\alpha + g \in \pi_n(M)$
(2). $F(\eta_{f'_\alpha}) = F(\eta_{f_\alpha}) + F(\eta_g)$ where F is the isomorphism:

$$\{n \text{ dimensional stable trivial vector bundles over } S^n\} \longleftrightarrow Ker(\pi_{n-1}(SO(n)) \to \pi_{n-1}(SO))$$

It is known that $Ker(\pi_{n-1}(SO(n)) \to \pi_{n-1}(SO)) = 0,\ n = 1,3,7$ and $Ker(\pi_{n-1}(SO(n)) \to \pi_{n-1}(SO)) = \mathbb{Z}_2$ with generator $F(TS^n)$, n odd $\neq 1,3,7$. ([2], p88).

In this case, modifying by this embedding g, we can make every element $\alpha \in K$ represented by an embedding with trivial normal bundle. So, for any free subgroup $\oplus_{i=1}^s(\mathbb{Z}\alpha_i \oplus \beta_i)$ of K with intersection matrix $\oplus_{i=1}^s \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$, M admits a decomposition:

$$M \cong M' \sharp\ s(S^n \times S^n)$$

These are the basic tools to removing (n dimensional) handles from a $2n$-manifold. In the next two sections, we will apply these tools to prove our main theorems.

3 Odd case

3.1 Wu class, quadratic function, and Kervaire invariant

Given a smooth manifold $(M^n, \partial M)$, the Steenrod operator $Sq = \sum_{i=0} Sq^i : H^*(M, \partial M, \mathbb{Z}_2) \to H^*(M, \partial M, \mathbb{Z}_2)$ determines a linear form on $H^*(M, \partial M, \mathbb{Z}_2)$:

$$H^*(M, \partial M) \longrightarrow \mathbb{Z}_2$$

$$x \mapsto < Sq(x), [M] >$$

where $[M] \in H_n(M, \partial M, \mathbb{Z}_2)$ is the fundamental class of the Poincaré pair $(M, \partial M)$. Since the cup product induces the isomorphism $H^*(M, \mathbb{Z}_2) \cong Hom(H^*(M, \partial M, \mathbb{Z}_2), \mathbb{Z}_2)$, there exists a unique element $v(M) = 1 + v_1(M) + v_2(M) + \cdots \in H^*(M, \mathbb{Z}_2)$ such that for each $x \in H^*(M, \partial M, \mathbb{Z}_2)$:

$$< v(M) \cup x, [M] >=< Sq(x), [M] >,\ < v_i(M) \cup x, [M] >=< Sq^i x, [M] >$$

Definition 3.1. *$v(M) = \sum_{i=0} v_i(M)$ is called the Wu class of M.*

By the definition, we see $v_i(M) = 0 \Longleftrightarrow Sq^i : H^{n-i}(M, \partial M, \mathbb{Z}_2) \to \mathbb{Z}_2$ is zero.

In his paper [3], Browder gave a geometric definition of Kervaire invariant, which is equivalent to his original definition of Kervaire invariant in [1]. This geometric definition is very close to the original definition of Kervaire, which is defined by the Arf invariant of a certain quadratic function, (cf [7]). First, it is known in [3] that:

Proposition 3.2. *For any $x \in H_n(M^{2n}, \mathbb{Z}_2)$, we can find an embedded $N^n \subset M^{2n}$ with $[N] = x$.*

Proposition 3.3. *If $M^{2n} \times \mathbb{R}^q \subset W^{2n+q}$, W connected and $y \in H_{n+1}(W, M, \mathbb{Z}_2)$, we can find $N \subset M$ representing $\partial y \in H_n(M, \mathbb{Z}_2)$, i.e $[N] = \partial y$ with $N = \partial V$, here $i : V \subset W \times [0,1]$ is a connected submanifold with $i_*[V] = y$, where $[V] \in H_{n+1}(V, \partial V, \mathbb{Z}_2)$ is the fundamental class. Furthermore, V meets $W \times 0$ transversally in $N \subset M$.*

We see, in this case, the normal bundle of N in $W \times 0$ has a normal q-frame ($N \times \mathbb{R}^q \subset M \times \mathbb{R}^q \subset W$). The obstruction to extending this frame to a normal q-frame on $V \subset W \times [0,1]$ lies in $H^{i+1}(V, N, \pi_i(V_{n+q,q}))$ and $\pi_i(V_{n+q,q}) = 0, q < n,\ \pi_n(V_{n+q,q}) = \mathbb{Z}_2$. We find the last and only one obstruction $\sigma \in H^{n+1}(V, N, \pi_n(V_{n+q,q})) = \mathbb{Z}_2$.

Definition 3.4. *For the element $x = \partial y \in H_n(M, \mathbb{Z}_2)$, where $\partial : H_{n+1}(W, M, \mathbb{Z}_2) \longrightarrow H_n(M, \mathbb{Z}_2)$ and $y \in H_{n+1}(W, M, \mathbb{Z}_2)$, we define: $\psi(x) =< \sigma, [V] >$, which is denoted briefly by σ for convenience.*

We see this definition seems not intrinsic, it depends on the choice of N and V. We should put some condition to make ψ well-defined. Browder proved ([3]):

Proposition 3.5. *The obstruction to extend a q-frame defines a quadratic form:*

$$\psi : Ker(H_n(M, \mathbb{Z}_2) \longrightarrow H_n(W, \mathbb{Z}_2)) \to \mathbb{Z}_2$$

if and only if $v_{n+1}(W) = 0$.

Proposition 3.6. *For the embedding $\phi(S^n) \in M^{2n}, n$ odd, if $\phi(S^n)$ is nullhomotopic in W, then $\psi([\phi(S^n)]) = 0$ if and only if the normal bundle of $\phi(S^n)$ is trivial.*

Definition 3.7. *If $Ker(H_n(M, \mathbb{Z}_2) \longrightarrow H_n(W, \mathbb{Z}_2))$ is non-singular under the intersection pair, we define the Kervaire invariant k by its Arf invariant of the quadratic form ψ.*

3.2 Proof of the odd case I

Let X^{n+1} be a projective toric manifold with complex dimension $n > 2$, odd, and $i : Y^n \hookrightarrow X^{n+1}$ be a hypersurface of X^{n+1}.

Lemma 3.8. *$H_n(Y, \mathbb{Z})$ is spherical and every element $\alpha \in H_n(Y, \mathbb{Z})$ can be represented by an embedding $f_\alpha : S^n \hookrightarrow Y$ such that the normal bundle η_{f_α} of f_α is stable trivial.*

Proof. First, by Lefschetz's hyperplane section theorem and Proposition 1.2., we know (X, Y) is n-connected and $H_n(X) = 0$. We have exact sequence:

$$\begin{array}{ccccc} H_{n+1}(X,Y) & \longrightarrow & H_n(Y) & \longrightarrow & H_n(X) = 0 \\ \uparrow \cong & & \uparrow h_Y & & \uparrow \\ \pi_{n+1}(X,Y) & \longrightarrow & \pi_n(Y) & \xrightarrow{\pi_n(i)} & \pi_n(X) \end{array}$$

From this diagram, we observe that $h_Y : \pi_n(Y) \longrightarrow H_n(Y)$ is surjective and for every element $\alpha \in H_n(Y)$, by the Whitney embedding theorem, we can choose an embedding $f_\alpha : S^n \hookrightarrow Y$ to represent α such that $i \circ f_\alpha$ is nullhomotopic in X, i.e. $\pi_n(i)([f_\alpha]) = 0$.

Second, we want to show the normal bundle η_{f_α} is stable trivial. We have bundle identity:

$$TX|S^n = (i \circ f_\alpha)^* TX = TS^n \oplus \eta_{f_\alpha} \oplus \eta_Y^X|S^n$$

here η_Y^X is the normal bundle of $i : Y \longrightarrow X$. Since η_Y^X is a complex line bundle, it is known that $\eta_Y^X \cong i^* L_Y$, where L_Y is a complex line bundle over X with Euler class $e(L_Y) \cap [X] = i_*[Y]$.

Since $i \circ f_\alpha$ is nullhomotopic, the bundle identity becomes:

$$\epsilon^{2n+2} = (i \circ f_\alpha)^* TX = TS^n \oplus \eta_{f_\alpha} \oplus (i \circ f_\alpha)^* L_Y = \epsilon^{n+1} \oplus \eta_{f_\alpha}$$

here ϵ is the trivial real 1-bundle. $\square$

Proof of the odd case I: For the complex line bundle L_Y in the above lemma, consider $W = D(-L_Y)$, where $-L_Y$ is the stable inverse bundle of L_Y, i.e. $L_Y \oplus -L_Y$ is trivial, and $D(-L_Y)$ is the disk bundle of $-L_Y$. Then for the embedding: $Y \hookrightarrow X \hookrightarrow W$, we see the normal bundle of Y in W is trivial and we get $Y \times \mathbb{R}^q \subset W$ for some $q > 0$.

Observe that $Ker(H_n(Y, \mathbb{Z}_2) \longrightarrow H_n(W, \mathbb{Z}_2)) = H_n(Y, \mathbb{Z}_2)$ and by proposition 3.5, if the Wu class $v_{n+1}(W) = 0$, there is a quadratic function $\psi' : H_n(Y, \mathbb{Z}_2) \longrightarrow \mathbb{Z}_2$ and we also obtain a quadratic function on $H_n(Y, \mathbb{Z})$:

$$\psi : H_n(Y, \mathbb{Z}) \longrightarrow H_n(Y, \mathbb{Z}_2) \xrightarrow{\psi'} \mathbb{Z}_2$$

Furthermore, by proposition 3.6, we know $\psi(\alpha) = 0$ if and only if the normal bundle η_{f_α} is trivial.

Since $H_n(Y, \mathbb{Z})$ is unimodular, by technique 2 in subsection 2.2, $H_n(Y, \mathbb{Z}) \cong \mathbb{Z}\alpha_0 \oplus \mathbb{Z}\beta_0 \oplus \oplus_{i=1}^s (\mathbb{Z}\alpha_i \oplus \beta_i)$, $s = \frac{b_n(Y)-2}{2}$ with intersection matrix $\oplus_{i=0}^s \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ and $\psi(\alpha_i) = \psi(\beta_i) = 0, i \neq 0$. Topologically, we get decomposition:

$$Y \cong Y' \sharp\ s(S^n \times S^n)$$

where $b_n(Y') = 2$. If the Kervaire (Arf) invariant k of ψ or ψ' vanishes, we can make $b_n(Y') = 0$.

So we finish the proof of the odd case when $v_{n+1}(W) = 0$. When $v_{n+1}(W) \neq 0$, the quadratic function ψ is not necessary well-defined and we will use technique 2 to deal with it in next subsection.

3.3 Proof of the odd case II

In his paper [3], Browder proved:

Theorem 3.9 (Browder). *Suppose $M^{2n} \times \mathbb{R}^q \subset W$, $n \neq 1, 3$ or 7, W is 1-connected. (W, M) is n-connected and suppose $v_{n+1}(W) \neq 0$. Then there exists an embedded $S^n \subset M^{2n}$ and $U^{n+1} \subset M^{2n} \times \mathbb{R}^{q+1}$ with $\partial U = S^n$ such that the normal bundle ξ to S^n in M^{2n} is nontrivial, but $\xi \oplus \epsilon^1$ is trivial, where ϵ^1 is the trivial one dimensional real vector bundle. Hence S^n is homologically trivial (mod 2) with nontrivial normal bundle.*

Remark 3.10. It seems we can use this theorem to find the embedding sphere in technique 2. But the shortage is: the embedding sphere S^n is only mod 2 trivial. We want to add some condition to make it work in integral homology.

Theorem 3.11. *Under the same hypothesis of Browder's theorem above, if we further assume that $H_{n+1}(W, \mathbb{Z}_2)$ is generated by the element $\{x_i\}$ such that each x_i can be represented by an oriented closed manifold N_i, i.e $[N_i] = x_i$. Then there exist an embedded $S^n \subset M^{2n}$ such that $[S^n] = 0 \in H_n(M, \mathbb{Z})$ and the normal bundle ξ to S^n in M^{2n} is non-trivial but stable trivial.*

Proof. We follow Browder's proof (Step 2 to Step 7 is almost unchanged):
Step 1: Since $v_{n+1}(W) \neq 0$, we know $Sq^{n+1} : H^{n+q-1}(W, \partial W, \mathbb{Z}_2) \to \mathbb{Z}_2$ is not zero. By assumption, $\exists N_i$ such that $Sq^{n+1}y_i \neq 0$, where $y_i \cap [W] = [N_i]$.
Step 2: For convenience, we denote N_i by N and y_i by y. Let $N_0 = N - intD^{n+1}$, we see $\partial N_0 = S^{n+1}$ and N_0 is homotopic to an n-complex. Since (W, M) is n-connected, $\exists f : N_0 \longrightarrow M$ such that the diagram is commutative up to homotopy:

$$\begin{array}{ccc} N_0 & \xrightarrow{f} & M \\ \downarrow & & \downarrow \\ N & \longrightarrow & W \end{array}$$

Step 3: Let $g = f|_{\partial N_0} : S^n \to M$, we see $g_*[S^n] = 0 \in H_n(M, \mathbb{Z})$. Since M is 1-connected, by Whitney's theorem, we can make g homotopic to an embedding and we still denote it by g. Since g is nullhomotopic in W, then the normal bundle of $g(S^n)$ is stable trivial. We wish to show that the normal bundle of this sphere is not trivial.
Step 4: We make the map $f : N_0 \longrightarrow M \times \mathbb{R}^q \times [-1, 0]$ homotopic to an embedding $g_0 : N_0 \hookrightarrow M \times \mathbb{R}^q \times [-1, 0]$ such that $g_0|_{\partial N_0} = g$. And we extend $g : S^n \hookrightarrow M$ to $\tilde{g} : D^{n+1} \subset W \times [0, 1]$ which meets $W \times 0$ transversally in $g(S^n)$. Then we get an embedding $g_1 : N_0 \cup_{S^n} D^{n+1} \cong N \hookrightarrow W \times [-1, 1]$, which is isotopic to the origin $N \subset W$.
Step 5: $M \times \mathbb{R}^q \subset W \times 0$ define a q-frame of the normal bundle of $g(S^n) \subset M \subset W \times 0$. We know the obstruction $\sigma \in \pi_{n+1}(V_{n+1,q})$ to extend this q-frame to D^{n+1} is zero if and only if the normal bundle of $g(S^n)$ is trivial.
Step 6: Now assume the normal bundle of $g(S^n)$ is trivial and we get a q-frame on D^{n+1}:

$$D^{n+1} \times D^n \times \mathbb{R}^q \subset W \times [0, 1]$$

such that $D^{n+1} \times 0 \times 0 = \tilde{g}(D)$ and $S^n \times D^n \times 0$ is the normal bundle of $g(S^n)$. Let $V = M \times [-1, 0] \cup_{S^n \times D^n} D^{n+1} \times D^n$, then we get $V \times \mathbb{R}^q \subset W \times [-1, 1]$, $g_1(N) \subset intV \times \mathbb{R}^q$.
Step 7: Let $Y = W \times [-1, 1]/\partial(W \times [-1, 1])$ we get:

$$Y \xrightarrow{a} \Sigma^q V/\partial V \xrightarrow{b} T(\eta_N \oplus \epsilon)$$

where η_N is the normal bundle of N in W. Let U be the mod 2 Thom class of $\eta_N \oplus \epsilon$, we get: $(ba)^*U = \Sigma x \in H^{n+q}(Y, \mathbb{Z}_2)$ and $(Sq^{n+1}(x))[W] = Sq^{n+1}(\Sigma x)[Y] \neq 0$. Also, $(Sq^{n+1}(b^*U))(\Sigma^q[V]) \neq 0$ and $Sq^{n+1}(\Sigma^{-q}(b^*U))[V] \neq 0$. On the other hand, $Sq^{n+1}(\Sigma^{-q}(b^*U)) = 0$ since $\Sigma^{-q}(b^*U)) = 0 \in H^n(V, \partial V, \mathbb{Z}_2)$. □

Proof of the odd case II: When $v_{n+1}(W) \neq 0$, in our case $Y \times \mathbb{R}^q \subset W = D(-L)$, by proposition 1.3, we see $H_*(X, \mathbb{Z})$ is generated by the toric submanifolds which are certainly oriented and $W = D(-L_Y)$ is the disk bundle over X, whose homology group is also generated by these toric submanifolds. Then all the conditions of theorem 3.11 are satisfied. Thus, there exists an embedding sphere $g : S^n \hookrightarrow Y$ such that $g_*[S^n] = 0$ and the normal bundle $\eta_g \cong TS^n$.

By the technique 2 in section 2 and lemma 3.8, we have topological decomposition:

$$Y \cong Y' \sharp\, s(S^n \times S^n)$$

where $b_n(Y') = 0$. Then we finish the proof of theorem 1.4.

4 Even case

4.1 Intersection form and signature

Let X^{n+1} be a projective toric manifold with complex dimension $n+1$, $n > 2$, even, and $i : Y \hookrightarrow X$ be a hypersurface of X. Since n is even, the n-th homology group $H_n(Y,\mathbb{Z})$ admits a unimodular symmetric intersection form:

$$H_n(Y,\mathbb{Z}) \otimes H_n(Y,\mathbb{Z}) \xrightarrow{\cdot} \mathbb{Z}$$

Since (X,Y) is n-connected and $H_{odd}(X,\mathbb{Z}) = 0$, like the odd case, we have

$$\begin{tikzcd}
0 \arrow[r] & H_{n+1}(X,Y) \arrow[r] & H_n(Y) \arrow[r, "i_*"] & H_n(X) \arrow[r] & 0 \\
& \pi_{n+1}(X,Y) \arrow[r] \arrow[u, "\cong"] & \pi_n(Y) \arrow[r, "\pi_n(i)"] \arrow[u, "h_Y"] & \pi_n(X) \arrow[u] &
\end{tikzcd}$$

The **vanishing cycles** $Ker(i_*) \subset H_n(Y,\mathbb{Z})$ is what we mainly concerned, because:

Lemma 4.1. *Each element $\alpha \in Ker(i_*)$ can be represented by an embedding $f_\alpha : S^n \hookrightarrow Y$ such that $f_\alpha[S^n] = \alpha$ and the normal bundle η_{f_α} of f_α is stable trivial.*

Proof. Since $\pi_n(X,Y) \cong H_n(X,Y,\mathbb{Z}) \cong Ker(i_*)$, we see for each element $\alpha \in Ker(i_*)$, there exists an embedding f_α representing α and $\pi_n(i)(f_\alpha) = 0 \in \pi_n(X)$.

The proof of the stable triviality of the normal bundle η_{f_α} is similar to the proof in lemma 3.8. $\square$

When we restrict the intersection form of $H_n(Y,\mathbb{Z})$ on $Ker(i_*)$, we get:

Proposition 4.2. *The intersection form on $Ker(i_*)$ is of type even, i.e. for any $\alpha \in Ker(i_*)$, $\alpha \cdot \alpha$ is even.*

Proof. For any $\alpha \in Ker(i_*)$, by lemma 4.1., we can use an embedding f_α to represent it. It is known that $\alpha \cdot \alpha =< e(\eta_{f_\alpha}), [S^n] >$, where $e(\eta_{f_\alpha})$ is the Euler class of the normal bundle η_{f_α}. Furthermore, $< e(\eta_{f_\alpha}), [S^n] >$ is even if and only if the n-th Stiefel-Whitney class $w_n(\eta_{f_\alpha})$ of η_{f_α} is zero and this is just proved in lemma 4.1. $\square$

The intersection pair on $H_n(Y,\mathbb{Z})$ is equivalent to the cup product on $H^n(Y,\mathbb{Z})$

$$H^n(Y,\mathbb{Z}) \otimes H^n(Y,\mathbb{Z}) \longrightarrow \mathbb{Z}$$

$$(\alpha, \beta) \mapsto < \alpha \cup \beta, [Y] >$$

through the Poincaré duality $PD : H^n(Y,\mathbb{Z}) \longrightarrow H_n(Y,\mathbb{Z})$ and we also have exact sequence:

$$0 \longrightarrow H^n(X,\mathbb{Z}) \xrightarrow{i^*} H^n(Y,\mathbb{Z}) \longrightarrow H^{n+1}(X,Y) \longrightarrow 0$$

We see the intersection form $(Ker(i_*), \cdot)$ is equivalent to $(PD^{-1}(Ker(i_*)), \cup)$ and the reason why we use the language of cohomolgy instead of homology is:

Lemma 4.3. $PD^{-1}(Ker(i_*)) = (i^*H^n(X))^{\perp}$

Proof. For any $\alpha \in Ker(i_*)$ and $\beta \in H^n(X,\mathbb{Z})$, we have:

$$< PD^{-1}(\alpha) \cup i^*\beta, [Y] >=< i^*\beta, \alpha >=< \beta, i_*\alpha >= 0$$

we get $PD^{-1}(Ker(i_*)) \subset (i^*H^n(X))^{\perp}$.

On the other hand, for any $PD^{-1}(\gamma) \in (i^*H^n(X))^{\perp}$, we see

$$< PD^{-1}(\gamma) \cup i^*H^n(X,\mathbb{Z}), [Y] >=< i^*H^n(X,\mathbb{Z}), \gamma >=< H^n(X,\mathbb{Z}), i_*\gamma >= 0$$

Since $H_n(X,\mathbb{Z})$ and $H^n(X,\mathbb{Z})$ are free Abel groups, we get $i_*\gamma = 0$. □

Next, we want to discuss some limit estimates about the n-th Betti number and the signature of the pair $(H^n(Y_d,\mathbb{Z}),\cup)$. Recall that $i_d : Y_d \hookrightarrow X^{n+1}$ is the hypersurface of the toric manifold X with $(i_d)_*[Y_d] = d(i_*[Y])$ and $degY_d =< \alpha_{Y_d}^{n+1}, [X] >= d^{n+1}degY$, where $\alpha_{Y_d} \cap [X] = (i_d)_*[Y_d]$. We have proposition:

Proposition 4.4. *We have limits:*

$$\lim_{d\to+\infty} \frac{b_n(Y_d)}{degY_d} = \lim_{d\to+\infty} \frac{b_n(Y_d)}{d^{n+1}degY} = 1$$

$$0 < \lim_{d\to+\infty} \frac{|sign(Y_d)|}{b_n(Y_d)} = 2^{n+2}(2^{n+2}-1)\frac{B_{\frac{n+2}{2}}}{(n+1)!} < 1$$

Proof. For the first limit, we know the Euler number $\chi(Y_d)$ of Y_d equals $b_n(Y_d)+2\sum_{j=1}^{n-1}(-1)^j b_j(X)$ and

$$\begin{aligned}\chi(Y_d) &= < c_n(Y_d), [Y_d] >=< \frac{c(TX)}{1+d\alpha_Y}, [X] > \\ &= d^{n+1} < \alpha_Y^{n+1}, [X] > +O(d^n)\end{aligned}$$

here $c(TX)$ and $c_n(Y_d)$ are the Chern classes. We have:

$$\lim_{d\to+\infty} \frac{\chi(Y_d)}{d^{n+1}} = \lim_{d\to+\infty} \frac{b_n(Y_d)}{d^{n+1}} = degY$$

and we get:

$$\lim_{d\to+\infty} \frac{b_n(Y_d)}{degY_d} = 1$$

For the second limit, we have identity:

$$sign(Y_d) =< \tanh(d\alpha_Y)L(X), [X] >$$

where $L(X) = L_1(X) + L_2(X) + \cdots$ is the L-class of X and $\tanh(d\alpha_Y) = \sum_{j=1}^{+\infty}(-1)^{j-1}2^{2j}(2^{2j} - 1)\frac{B_j}{(2j)!}(d\alpha_Y)^{2j-1}$. Observe that:

$$sign(Y_d) = (-1)^{\frac{n}{2}}2^{n+2}(2^{n+2}-1)\frac{B_{\frac{n+2}{2}}}{(n+2)!}d^{n+1}degY + O(d^n)$$

we have limit:

$$\lim_{d\to+\infty}\frac{|sign(Y_d)|}{b_n(Y_d)}=2^{n+2}(2^{n+2}-1)\frac{B_{\frac{n+2}{2}}}{(n+1)!}$$

Furthermore, when $j>1$, we see:

$$1+\frac{1}{2^{2j}}+\frac{1}{3^{3j}}+\cdots=\frac{B_j(2\pi)^{2j}}{2(2j)!}<\frac{\pi^2}{6}$$

$$\frac{B_j2^{2j}(2^{2j}-1)}{(2j)!}<\frac{\pi^2}{3}\frac{2^{2j}(2^{2j}-1)}{(2\pi)^{2j}}<\frac{\pi^2}{3}\frac{4^j}{\pi^{2j}}<1$$

$\square$

Corollary 4.5. $\lim_{d\to+\infty}b_n(Y_d)=+\infty$ *and* $\lim_{d\to+\infty}sign(Y_d)=+\infty$*. When d is big enough, $(H_n(Y_d,\mathbb{Z}),\cdot)$ is indefinite.*

4.2 Proof of the even case

Let $(H,<,>)$ be a unimodular symmetric bilinear form over $\mathbb{Z}$ and F be a nonzero subgroup of H such that H/F is free and the map $F\longrightarrow Hom(F,\mathbb{Z})$ induced by $<,>$ is injective. Denote $E=F^{\perp}:=\{x\in H|<x,F>=0\}$, we have:

Theorem 4.6. *If $rankH\geqslant Max\{4rankF,2rankF+5\}$, then E admits an orthogonal decomposition:*

$$(E,<,>)\cong(A,<,>)\oplus(\oplus_{i=1}^{s}(\mathbb{Z}x_i\oplus\mathbb{Z}y_i,<,>))$$

where the intersection matrix of $\mathbb{Z}x_i\oplus\mathbb{Z}y_i$ is $\begin{pmatrix}0&1\\1&c_i\end{pmatrix}$, $c_i=0$ or 1. For $(A,<,>)$, there are two possibilities:
(1). $(A,<,>)$ is definite and $rankA\geqslant max\{3rankF,rankF+5\}$
(2). $rankA<max\{3rankF,rankF+5\}$

We'll prove the even case first and the proof of this theorem will be given in the next subsection.

Proof of the even case: **Step 1:** For the bilinear symmetric space $(H^n(Y_d,\mathbb{Z}),\cup)$, we know $PD^{-1}(Ker((i_d)_*))=(i^*H^n(X))^{\perp}$. We want to show the injectivity of the map $H^n(X,\mathbb{Z})\longrightarrow Hom(H^n(X,\mathbb{Z}),\mathbb{Z})$ induced by the cup product in $H^n(Y_d,\mathbb{Z})$.

Since Y_d is the hypersurface of X, the hard Lefschetz theorem ([11]) tell us that the cohomology element α_{Y_d} representing Y_d induces an injective map:

$$\cup\alpha_{Y_d}:H^n(X,\mathbb{Z})\longrightarrow H^{n+2}(X,\mathbb{Z})$$

For $i^*H^n(X,\mathbb{Z})\subset H^n(Y_d,\mathbb{Z})$, we have diagram:

$$\begin{array}{ccccc}
H^n(X,\mathbb{Z}) & \xrightarrow{\cup\alpha_{Y_d}} & H^{n+2}(X,\mathbb{Z}) & \xrightarrow{\cong} & Hom(H^n(X,\mathbb{Z}),\mathbb{Z})\\
{\scriptstyle i_d^*}\big\downarrow & & & & \big\uparrow\\
i_d^*H^n(X,\mathbb{Z}) & \longrightarrow & H^n(Y_d,\mathbb{Z}) & \xrightarrow{\cong} & Hom(H^n(Y_d,\mathbb{Z}),\mathbb{Z})
\end{array}$$

indeed, for any $x, y \in H^n(X,\mathbb{Z})$, $x(y) =< i^*x \cup i^*y, [Y_d] >=< x \cup y \cup \alpha_{Y_d}, [X] >= (x \cup \alpha_{Y_d})(y)$.

Furthermore, we see the restriction of $(H^n(Y_d,\mathbb{Z}),\cup)$ to $H^n(X,\mathbb{Z})$ is just the bilinear form defined in theorem 1.5.

Thus we get a pair $(H_d,\cup) = (H^n(Y_d,\mathbb{Z}),\cup)$ with a free subgroup $F := i^*H^n(X,\mathbb{Z})$ such that

(1). $F^\perp = PD^{-1}(Ker(i_d)_*) = E_d$ with even type (proposition 4.2, lemma 4.3).

(2). If d is big enough, $rankH_d > Max\{4rankF, 2rankF + 5\}$ (proposition 4.4)

Step 2: By the algebraic decomposition theorem 4.6,

$$(Ker(i_d)_*, \cdot) \cong (E_d,\cup) \cong A_d \oplus (\oplus_{i=1}^{s_d}(\mathbb{Z}x_i \oplus \mathbb{Z}y_i, <,>))$$

where the intersection matrix of $\mathbb{Z}x_i \oplus \mathbb{Z}y_i$ is $\begin{pmatrix} 0 & 1 \\ 1 & c_i \end{pmatrix}$, $c_i = 0$ or 1

By proposition 4.2, $Ker((i_d)_*)$ is of type even, c_i must be zero. Since $\lim_{d\to\infty} |signH^n(Y_d,\mathbb{Z})| = +\infty$, when d is big enough, the possibility (2) of theorem 4.6 can not happen, and A_d is definite.

Step 3: By the process of removing handles of the even case in section 2, we see

$$Ker(i_d)_* = A_d \oplus \oplus_{i=1}^{s_d}(\mathbb{Z}x_i \oplus \mathbb{Z}y_i)$$

where the intersection matrix of $\oplus_{i=1}^{s_d}(\mathbb{Z}x_i \oplus \mathbb{Z}y_i)$ is $\oplus_{s_d}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ and we get:

$$Y_d \cong Y_d' \sharp s_d(S^n \times S^n)$$

Since A_d is definite and $sign(\mathbb{Z}x_i \oplus \mathbb{Z}y_i) = 0$, we get identiy $2s_d = rank(Ker(i_d)_*) - |sign(Ker(i_d)_*)|$. Also, since $Ker(i_d)_* = (H^n(X,\mathbb{Z}))^\perp \subset H^n(Y_d,\mathbb{Z})$ and the restriction of $(H^n(Y_d,\mathbb{Z}),\cup)$ to $H^n(X,\mathbb{Z})$ is just the bilinear form defined in theorem 1.5, we get

$$2s_d = b_n(Y_d) - b_n(X) - |sign(Y_d) - sign(H^n(X))|$$

For the limit estimate, we have:

$$\lim_{d\to+\infty} \frac{2s_d}{degY_d} = \lim_{d\to+\infty} \frac{2s_d}{b_n(Y_d)} = 1 - \lim_{d\to+\infty} \frac{|sign(Y_d)|}{b_n(Y_d)}$$

4.3 Proof of the algebraic decomposition theorem

In order to prove theorem 4.6., we need some lemmas.

Lemma 4.7. *Assume E satisfy $rankE \geqslant 3rankF$, we can choose a basis $\{f_1, f_2, \cdots, f_{r+h}\}$ of $Hom(E,\mathbb{Z})$ such that $\oplus_{i=1}^r \mathbb{Z}f_i \to Hom(E,\mathbb{Z})/E$ is surjective, $r \leqslant rankF$, and $\oplus_{j=1}^h \mathbb{Z}f_{r+j} \subset E \subset Hom(E,\mathbb{Z})$, $h \geqslant 2r$.*

Proof. First, since H/F is free, we have:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & E & \xrightarrow{<,>} & Hom(H,\mathbb{Z}) & \longrightarrow & Hom(F,\mathbb{Z}) & \longrightarrow & 0 \\ & & \| & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & E & \xrightarrow{<,>} & Hom(E,\mathbb{Z}) & \longrightarrow & Hom(E,\mathbb{Z})/E & \longrightarrow & 0 \end{array}$$

Since $H/E \cong Hom(H,\mathbb{Z})/E \cong Hom(F,\mathbb{Z})$, we see $Hom(H,\mathbb{Z}) \longrightarrow Hom(E,\mathbb{Z})$ is surjective and $Hom(F,\mathbb{Z}) \longrightarrow Hom(E,\mathbb{Z})/E$ is also surjective.

Second, $rankHom(F,\mathbb{Z}) = rankF$, we can choose $rankF$ elements $\{g_1, g_2 \cdots\}$ of $Hom(E,\mathbb{Z})$ such that $\mathbb{Z}g_1 + \mathbb{Z}g_2 + \cdots + \mathbb{Z}g_{rankF} \longrightarrow Hom(E,\mathbb{Z})/E$ is surjective. Then there is a subgroup $\mathbb{Z}g_1 + \mathbb{Z}g_2 + \cdots + \mathbb{Z}g_{rankF} \subset N \subset Hom(E,\mathbb{Z})$ with $Hom(E,\mathbb{Z})/N$ free and $r = rankN = rank(\mathbb{Z}g_1 + \mathbb{Z}g_2 + \cdots + \mathbb{Z}g_{rankF}) \leqslant rankF$.

Third, Since $Hom(E,\mathbb{Z})/N$ is free, let $\{f_1, \cdots, f_r\}$ be a basis of N and extend it to a basis $\{f_1, \cdots, f_r, f'_{r+1}, \cdots, f'_{r+h}\}$ of $Hom(E,\mathbb{Z})$. We know $N \longrightarrow Hom(E,\mathbb{Z})/E$ is surjective, then for any f'_{r+i}, we can find $f_{r+i} = f'_{r+i} - \sum_{j=1}^{r} a_{ij} f_j$ with $[\sum_{j=1}^{r} a_{ij} f_j] = [f'_{r+i}] \in Hom(E,\mathbb{Z})/E$, we see $f_{r+i} \in E$.

Thus we obtain a basis $\{f_1 \cdots f_r, f_{r+1}, \cdots f_{r+h}\}$ of $Hom(E,\mathbb{Z})$ such that $\{f_{r+1}, \cdots f_{r+h}\} \subset E \subset Hom(E,\mathbb{Z})$. □

Lemma 4.8. *Assume E is indefinite and $rankE \geqslant Max\{3rankF, rankF + 5\}$, we can find two elements $x, y \in E$ with $<x,x>= 0$, $<x,y>= 1, <y,y>= 0$ or 1.*

Proof. First, by the lemma above, we have a basis $\{f_1 \cdots f_r, f_{r+1}, \cdots f_{r+h}\}$ of $Hom(E,\mathbb{Z})$ with $\mathbb{Z}f_{r+1} \oplus \mathbb{Z}f_{r+2} \oplus \cdots \oplus \mathbb{Z}f_{r+h} \subset E \subset Hom(E,\mathbb{Z})$, $h > max\{2r, 5\}$. Let $\{f_1^*, \cdots f_{r+s}^*\}$ be the dual of the basis $\{f_1, \cdots f_{r+h}\}$ in $Hom(E,\mathbb{Z})^* = E$. Define:

$$D = \mathbb{Z}f_{r+1}^* \oplus \cdots \mathbb{Z}f_{r+h}^* \subset E = Hom(E,\mathbb{Z})^*$$

Second, when D is indefinite, since $rankD \geqslant 5$, it is known from Meyer's theorem that there exists an indivisible element $x \in D$ such that $<x,x>= 0$ ([8], p255). Then we can also choose an element $y' \in \mathbb{Z}f_{r+1} \oplus \cdots \oplus \mathbb{Z}f_{r+h} \subset E$ such that $<x,y'>= 1$. Let $y = y' - [\frac{<y',y'>}{2}]x'$, we have $<x,x>= 0, <x,y>= 1, <y,y>= 0$ or 1.

Third, when D happens to be definite, define $D' := \mathbb{Z}(f_{r+1}^* - c_1 f_1^*) \oplus \mathbb{Z}(f_{r+2}^* - c_2 f_1^*) \oplus \mathbb{Z}(f_{r+3}^* - c_3 f_2^*) \oplus \mathbb{Z}(f_{r+4}^* - c_4 f_2^*) \oplus \cdots \oplus \mathbb{Z}(f_{3r-1}^* - c_{2r-1} f_r^*) \oplus \mathbb{Z}(f_{3r}^* - c_{2r} f_r^*) \oplus \cdots \oplus \mathbb{Z}(f_{r+h}^* - c_h f_r^*)$.

If we can choose proper $\{c_i \in \mathbb{Z}\}$ to make D' indefinite, we can still find an indivisible element $x \in D'$ such that $<x,x>= 0$, and we can also find $y \in \oplus_{j=1}^{h} \mathbb{Z}f_{r+j} \in E$ such that $<x,y>= 1$. So, all we need to do is to prove the next lemma. □

Lemma 4.9. *Following lemma 4.8, suppose D is definite, we can choose proper $\{c_i \in \mathbb{Z}\}$ to make D' indefinite.*

Proof. Assume D is positive definite under $<,>$. Consider the real space $\overline{E} := E \otimes \mathbb{R}$, $\overline{D} := D \otimes \mathbb{R}$, $\overline{D'} := D' \otimes \mathbb{R}$ and let $\{f_1^*, \cdots, f_{r+h}^*\}$ be the Euclidean orthogonal standard basis of $\overline{E}$. Define:

$$F : \overline{E} \longrightarrow \mathbb{R}$$

$$\sum a_i f_i^* \mapsto \sum a_i a_j < f_i^*, f_j^* >$$

Observe that F is just the the extension of the map $: E \to \mathbb{Z}, x \mapsto <x,x>$ to $\overline{E}$.

Note that E is indefinite and $\mathbb{Q}$-uninodular under $<,>$, since by assumption H is unimodular and F is $\mathbb{Q}$-unimodular. Then we can find a $v \in \overline{E}$ such that $F(v) < 0$ and the Euclidean norm

$|v| = 1$, i.e. $v = \sum_{i=1}^{r} a_i f_i^* + \sum_{j=1}^{h} b_j f_{r+j}^*$, $\sum a_i^2 + \sum b_j^2 = 1$. Since D is definite, we see $F(\overline{D} - \{0\}) > 0$ and $(a_1, \cdots, a_r) \neq (0, \cdots, 0)$.

In the Euclidean norm with orthogonal standard basis $\{f_i^*\}$, we have decomposition $\overline{E} = \overline{D'} \oplus (\overline{D'})^{\perp}$. By calculation, we see $(\overline{D'})^{\perp}$ has a standard orthogonal basis:

$$(\overline{D'})^{\perp} = span\{\frac{f_1^* + c_1 f_{r+1}^* + c_2 f_{r+2}^*}{\sqrt{1+c_1^2+c_2^2}}, \frac{f_2^* + c_3 f_{r+3}^* + c_4 f_{r+4}^*}{\sqrt{1+c_3^2+c_4^2}}$$

$$\cdots, \frac{f_{r-1}^* + c_{2r-3} f_{3r-3}^* + c_{2r-2} f_{3r-2}^*}{\sqrt{1+c_{2r-3}^2+c_{2r-2}^2}}, \frac{f_r^* + c_{2r-1} f_{3r-1}^* + c_{2r} f_{3r}^* + \cdots, c_h f_{r+s}^*}{\sqrt{1+c_{2r-1}^2+\cdots+c_h^2}}\}$$

For convenience, denote this basis by $\{g_1, g_2, \cdots, g_r\}$.

For the vector $v = \sum_{i=1}^{r} a_i f_i^* + \sum_{j=1}^{h} b_j f_{r+j}^*$, we can decompose $v = v_1 + v_2$ such that $v_1 \in \overline{D'}$ and $v_2 \in (\overline{D'})^{\perp}$. By calculation,

$$v_2 = \frac{a_1 + c_1 b_{r+1} + c_2 b_{r+2}}{\sqrt{1+c_1^2+c_2^2}} g_1 + \cdots + \frac{a_{r-1} + c_{2r-3} b_{3r-3} + c_{2r-2} b_{3r-2}}{\sqrt{1+c_{2r-3}^2+c_{2r-2}^2}} g_{r-1}$$

$$+ \frac{a_r + c_{2r-1} b_{3r-1} + c_{2r} b_{3r} + \cdots, c_h b_{r+h}}{\sqrt{1+c_{2r-1}^2+\cdots+c_h^2}} g_r$$

Since $\sum a_i^2 + \sum b_j^2 = 1$, for $\forall \epsilon > 0$, we can choose proper $c_i \in \mathbb{Z}$ ([8], p256) such that

$$|\frac{a_1 + c_1 b_{r+1} + c_2 b_{r+2}}{\sqrt{1+c_1^2+c_2^2}}| < \frac{\epsilon}{r}, \cdots, |\frac{a_{r-1} + c_{2r-3} b_{3r-3} + c_{2r-2} b_{3r-2}}{\sqrt{1+c_{2r-3}^2+c_{2r-2}^2}}| < \frac{\epsilon}{r}$$

$$|\frac{a_r + c_{2r-1} b_{3r-1} + c_{2r} b_{3r} + \cdots, c_h b_{r+h}}{\sqrt{1+c_{2r-1}^2+\cdots+c_h^2}}| < \frac{\epsilon}{r}$$

The function F is continuous and $F(v) < 0$, if the Euclidean norm of $v_2 = v - v_1$ is small enough, then the element $v_1 \in \overline{D'}$ satisfy $F(v_1) < 0$. Furthermore, D' is not negative definite, since D is positive and $rankD' = rankD, 2rankD > rankE = rankD + rankF$, thus we see D' is indefinite.

$\square$

Proof of theorem 4.6.: We use induction on $rankH$, since $rankH \geqslant Max\{4rankF,\ 2rankF + 5\}$, we get $rankE \geqslant Max\{3rankF, rankF + 5\}$. If E is definite, we've done. If E is indefinite, then by the lemmas we've just proved, there exist two elements $x, y \in E$ such that $< x, x >= 0, < x, y >= 1, < y, y >= 0$ or 1.

We get orthogonal decomposition under $<,>$:

$$H = H' \oplus (\mathbb{Z}x \oplus \mathbb{Z}y),\ E = E' \oplus (\mathbb{Z}x \oplus \mathbb{Z}y)$$

Observe that $F \subset H'$ and $E \cap H' = E' = F^{\perp} \subset H'$, by the induction, we've finished our proof.

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